

# Complementary Dual Algebraic Geometry Codes

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## Abstract

Linear complementary dual (LCD) codes is a class of linear codes introduced by Massey in 1964. LCD codes have been extensively studied in literature recently. In addition to their applications in data storage, communications systems, and consumer electronics, LCD codes have been employed in cryptography. More specifically, it has been shown that LCD codes can also help improve the security of the information processed by sensitive devices, especially against so-called side-channel attacks (SCA) and fault non-invasive attacks. In this paper, we are interested in the construction of particular algebraic geometry (AG) LCD codes which could be good candidates to be resistant against SCA. We firstly provide a construction scheme for obtaining LCD codes from elliptic curves. Then, some explicit LCD codes from elliptic curve are presented. MDS codes are of the most importance in coding theory due to their theoretical significance and practical interests. In this paper, all the constructed LCD codes from elliptic curves are MDS or almost MDS. Some infinite classes of LCD codes from elliptic curves are optimal due to the Griesmer bound. Finally, we introduce a construction mechanism for obtaining LCD codes from any algebraic curve and derive some explicit LCD codes from hyperelliptic curves and Hermitian curves.

## Index Terms

Linear complementary dual codes, algebraic geometry codes, algebraic curves, elliptic curves, non-special divisors

## I. INTRODUCTION

Linear complementary dual (LCD) cyclic codes over finite fields were first introduced and studied by Massey [15] in 1964. In the literature LCD cyclic codes were referred to as reversible cyclic codes. It is well-known that LCD codes are asymptotically good. Furthermore, using the full dimension spectra of linear codes, Sendrier showed that LCD codes meet the asymptotic Gilbert-Varshamov bound [21]. Afterwards, LCD codes have been extensively studied in literature. In particular many properties and constructions of LCD codes have been obtained. Yang and Massey have provided in [24] a necessary and sufficient condition under which a cyclic code have a complementary dual. Dougherty et al. have developed in [6] a linear programming bound on the largest size of a LCD code of given length and minimum distance. Esmaeili and Yari analyzed LCD codes that are quasi-cyclic [7]. Muttoo and Lal constructed a reversible code over  $\mathbb{F}_q$  [18]. Tzeng and Hartmann proved that the minimum distance of a class of reversible cyclic codes is greater than the BCH bound [19]. In [13] Li et al. studied a class of reversible BCH codes proposed in [12] and extended the results on their parameters. As a byproduct, the parameters of some primitive BCH codes have been analyzed. Some of the obtained codes are optimal or have the best known parameters. In [3] Carlet and Guilley investigated an application of LCD codes against side-channel attacks, and presented several constructions of LCD codes. In [5], Ding et al. constructed several families of LCD cyclic codes over finite fields and analyzed their parameters.

Let  $K = \mathbb{F}_q$  be a finite field of order  $q$  and  $C/K$  be a smooth projective curve of genus  $g$ . We denote by  $D$  a divisor over  $C/K : D := P_1 + \dots + P_n$ , where  $P_i (i = 1, \dots, n)$  are pairwise different places of degree one. Let  $G$  be a divisor of  $C/K$  such that  $\text{supp } D \cap \text{supp } G = \emptyset$ . Let  $\mathcal{C} := \mathcal{C}(D, G)$  be the associate algebraic geometry (AG) code with the divisors  $D$  and  $G$  defined as

$$\mathcal{C}(D, G) = \{(f(P_1), \dots, f(P_n)), f \in \mathcal{L}(G)\},$$

where  $\mathcal{L}(G) = \{f \in K(C), (f) \succeq -G\} \cup \{0\}$ . The code  $\mathcal{C}$  is the image of  $\mathcal{L}(G)$  under the evaluation map  $ev_D$  given by

$$\begin{aligned} ev_D : \mathcal{L}(G) &\longrightarrow \mathbb{F}_q^n \\ f &\longmapsto (f(P_1), \dots, f(P_n)). \end{aligned}$$

An algebraic geometry (AG) code  $\mathcal{C}(D, G)$  associating with divisors  $G$  and  $D$  over the projective line is said to be *rational*. In particular BCH codes and Goppa codes can be described by means of rational AG codes. All the generalized Reed-Solomon codes and extended generalized Reed-Solomon codes can be defined under the framework of AG codes.

Recently, it has been shown that codes can also help improve the security of the information processed by sensitive devices, especially against the so-called *side-channel attacks* (SCA) and fault non-invasive attacks.

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In this paper, we are interested in the construction of particular AG complementary dual (LCD) codes which can be resistant against SCA. We firstly provide a construction scheme for obtaining complementary dual codes from elliptic curves (Theorem 3.2). Then, some explicit complementary dual code are presented. All the constructed LCD codes from elliptic curve are MDS or almost MDS. Moreover, they contain some infinite class of optimal codes meeting Griesmer bound on linear codes. Finally, we introduce a construction mechanism for obtaining LCD codes from any algebraic curve (Theorem 5.1 and Theorem 5.2) and give some explicit LCD codes from hyperelliptic curve and Hermitian curves. The constructed LCD codes presented in this paper could be good candidates of codes resistant against SCA, that is, codes having the property of being complementary dual codes with high minimal distance closed the Singleton bound.

This paper is organized as follows: In Section II we introduce the notations used in this paper and recall some basic facts about algebraic geometry codes. In section III, we present a general construction of LCD codes from elliptic curves. In Section IV, some explicit LCD codes from elliptic curves are derived. In Section V, we introduce a construction mechanism for obtaining LCD codes from any algebraic curves and give some explicit LCD codes from hyperelliptic curve and Hermitian curves.

## II. PRELIMINARIES

In this section, we introduce notations and results on LCD codes, algebraic geometry codes, and elliptic curves.

### A. Complementary dual codes and optimal codes

A linear code of length  $n$  over  $\mathbb{F}_q$  is a linear subspace of  $\mathbb{F}_q^n$ . There is a canonical non-degenerate bilinear form on  $\mathbb{F}_q^n \times \mathbb{F}_q^n$ , defined by

$$\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle = a_1 b_1 + \dots + a_n b_n.$$

For a linear code  $\mathcal{C}$  of length  $n$ , the code

$$\mathcal{C}^\perp = \{\mathbf{v} \in \mathbb{F}_q^n : \langle \mathbf{v}, \mathbf{c} \rangle = 0 \text{ for any } \mathbf{c} \in \mathcal{C}\}$$

is called the dual of  $\mathcal{C}$ . The code  $\mathcal{C}^\perp$  is linear, and we have

$$\dim_{\mathbb{F}_q}(\mathcal{C}) + \dim_{\mathbb{F}_q}(\mathcal{C}^\perp) = n.$$

A linear code  $\mathcal{C}$  is said to be a *linear complementary dual (LCD) code* if the intersection with its dual  $\mathcal{C}^\perp$  is trivial, that is,  $\mathcal{C} \cap \mathcal{C}^\perp = \{0\}$ . The weight  $wt(\mathbf{v})$  of a vector  $\mathbf{v} \in \mathbb{F}_q^n$  is the number of its nonzero coordinates. The minimum Hamming distance  $d$  of a linear code  $\mathcal{C} \neq \{0\}$  is defined by

$$d = \min\{wt(\mathbf{c}) : \mathbf{c} \in \mathcal{C}\}.$$

An  $[n, k, d]$  linear code  $\mathcal{C}$  is a linear code of length  $n$ , dimension  $k$  and minimum Hamming distance  $d$ . We shall use the following codes.

**Definition** Let  $\mathbf{a} = (a_1, \dots, a_n)$  with  $a_i \in \mathbb{F}_q^*$  and  $\mathcal{C} \subseteq \mathbb{F}_q^n$ . Then

$$\mathbf{a} \cdot \mathcal{C} := \{(a_1 c_1, \dots, a_n c_n) : (c_1, \dots, c_n) \in \mathcal{C}\}.$$

Obviously,  $\mathbf{a} \cdot \mathcal{C}$  is a linear code if and only if  $\mathcal{C}$  is a linear code. These codes have the same dimension, minimum Hamming distance and weight distribution.

Let  $n_q(k, d) := \min\{n : \text{there is an } [n, k, d] \text{ linear code over } \mathbb{F}_q\}$ . The  $[n_q(k, d), k, d]$  codes are called *optimal codes*. The following result is known as the *Griesmer bound* (see [4] or [10]).

$$n_q(k, d) \geq g_q(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil. \quad (1)$$

Under certain conditions on  $d$  and  $q$ , it can be shown that  $n_q(k, d) = g_q(k, d)$  (see [4] and [9]) for further references. Any  $[g_q(k, d), k, d]$  code is optimal. Obviously,

$$n_q(k, d) \geq g_q(k, d) \geq k + d - 1.$$

The inequality  $n_q(k, d) \geq k + d - 1$  is known as the *Singleton bound*. If  $d > q$ , then the Singleton bound is always worse than the Griesmer bound. The  $[n, k, d]$  codes with  $n = k + d - 1$  (resp.  $n = k + d$ ) are called *maximum distance separable (MDS) codes* (resp. *almost maximum distance separable (MDS) codes*).

## B. Generalized algebraic geometry codes

Let  $C$  be a smooth projective curve of genus  $g$ . Throughout this paper, we assume  $P_1, P_2, \dots, P_n$  are pairwise different places of  $C$  of degree one and denote by  $D$  the divisor  $P_1 + P_2 + \dots + P_n$ . We fix some notations which will be used throughout this paper.

- $\mathbb{F}_q$  denotes the finite field with  $q = p^m$  elements;
- $\text{Tr}_1^m(x) = \sum_{i=0}^{m-1} x^{p^i}$  denotes the trace function from  $\mathbb{F}_q$  to  $\mathbb{F}_p$ ;
- $C$  denotes a smooth projective curve over  $\mathbb{F}_q$ ;
- $\mathbb{F}_q(C)$  denotes the function field of  $C$ ;
- $C(\mathbb{F}_q)$  denotes the set of  $\mathbb{F}_q$ -rational points of  $C$ ;
- $\Omega$  denotes the module of differentials of  $C$ ;
- $(f)$  denotes the principal divisor of  $0 \neq f \in \mathbb{F}_q(C)$ ;
- $(\omega)$  denotes the divisor of differential  $0 \neq \omega \in \Omega$ ;
- $v_P$  denotes the valuation of  $\mathbb{F}_q(C)$  at the place  $P$ ;
- $\text{Res}_P(\omega)$  denotes the residue of  $\omega$  at  $P$ ;
- $G$  denotes a divisor of  $C$  over  $\mathbb{F}_q$ ;
- $\text{Supp}(G)$  denotes the set of places in the support of  $G$ ;
- $\mathcal{L}(G) := \{f \in \mathbb{F}_q(C) : (f) \succeq -G\} \cup \{0\}$ ;
- $\Omega(G) := \{\omega \in \Omega : (\omega) \succeq G\} \cup \{0\}$ ;
- $l(G)$  denotes the dimension of  $\mathcal{L}(G)$  over  $\mathbb{F}_q$ ;
- $i(G)$  denotes the dimension of  $\Omega(G)$  over  $\mathbb{F}_q$ .

Two divisor  $D_1$  and  $D_2$  are called equivalent, if there is a function  $f \in \mathbb{F}_q(C)$  with  $(f) = D_1 - D_2$ . Denote two equivalent divisors  $D_1$  and  $D_2$  by  $D_1 \sim D_2$ . The following famous result [23], known as the Riemann-Roch theorem is not only a central result in algebraic geometry with applications in other areas, but it is also the key of several results in coding theory.

*Theorem 2.1:* Let  $G$  be a divisor on a smooth projective curve of genus  $g$  over  $\mathbb{F}_q$ . Then, for any Weil differential  $0 \neq \omega \in \Omega$

$$l(G) - i(G) = \deg(G) + 1 - g \quad \text{and} \quad i(G) = l((\omega) - G).$$

We call  $i(G)$  the index of speciality of  $G$ . A divisor  $G$  is called *non-special* if  $i(G) = 0$  and otherwise it is called *special*. Note that  $g - 1$  is the least possible degree of a divisor of  $G$  to be non-special, since  $0 \leq l(G) = \deg(G) - g + 1$ . Moreover, if  $\deg(G) = g - 1$ , then  $G$  is a non-special divisor if and only if  $l(G) = 0$ . A non-special divisor of degree  $g - 1$  is never effective.

Let  $G = \sum_{i=1}^n m_i P_i$  and  $H = \sum_{i=1}^n m'_i P_i$  be two divisors. Then, we call  $\sum_{i=1}^n \min(m_i, m'_i) P_i$  the greatest common divisors denoted by  $\text{g.c.d}(G, H)$ . Such a divisor is supported on the places common to the support of both divisors with coefficients the minimum of those occurring in  $G$  and  $H$ . We call  $\sum_{i=1}^n \max(m_i, m'_i) P_i$  the least multiple divisor denoted by  $\text{l.m.d}(G, H)$ . Such a divisor is supported on all the places in the supports of  $G$  and  $H$  with coefficients the maximum of those occurring in the divisors  $G$  and  $H$ .

For a divisor  $G$  of  $C$  with  $v_{P_i}(G) = 0 (i = 1, \dots, n)$  and  $2g - 2 < \deg(G) < n$ , and a vector  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  with  $a_i \in \mathbb{F}_q^*$ , we define a *generalized algebraic geometry code*

$$\mathcal{GC}(D, G, \mathbf{a}) := \{(a_1 f(P_1), \dots, a_n f(P_n)) : f \in \mathcal{L}(G)\}. \quad (2)$$

If  $\mathbf{a} = (1, 1, \dots, 1)$ , then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a classical algebraic geometry code denoted by  $\mathcal{C}(D, G)$ . If  $C$  is a curves of genus 1 (called *elliptic curves*),  $\mathcal{GC}(D, G, \mathbf{a})$  (resp.  $\mathcal{C}(D, G)$ ) is called *generalized elliptic code* (resp. *elliptic code*).

Let  $\omega$  be a *Weil differential* such that  $v_{P_i}(\omega) = -1$  for  $i \in \{1, \dots, n\}$ . Then  $\mathcal{C}(D, G)^\perp = \mathbf{e} \cdot \mathcal{C}(D, H)$  with  $H := D - G + (\omega)$  and  $\mathbf{e} = (\text{Res}_{P_1}(\omega), \dots, \text{Res}_{P_n}(\omega))$ . Thus,

$$\mathcal{GC}(D, G, \mathbf{a})^\perp = (\mathbf{a}^{-1} * \mathbf{e}) \cdot \mathcal{C}(D, H), \quad (3)$$

where  $\mathbf{a}^{-1} * \mathbf{e} = (\frac{\text{Res}_{P_1}(\omega)}{a_1}, \dots, \frac{\text{Res}_{P_n}(\omega)}{a_n})$ .

The following theorem determines the parameters of  $\mathcal{GC}(D, G, \mathbf{a})$  [23].

*Theorem 2.2:* The code  $\mathcal{GC}(D, G, \mathbf{a})$  has dimension  $k = \deg(G) - g + 1$  and minimum distance  $d \geq n - \deg(G)$ .

From the definition of the generalized algebraic geometry codes we see that curves carrying many rational points may produce long codes. On the other hand, the number of  $\mathbb{F}_q$ -rational points of a smooth curve  $C$  defined over  $\mathbb{F}_q$  is bounded by the well known Hasse-Weil bound:

$$|\#(C(\mathbb{F}_q)) - (q + 1)| \leq 2g\sqrt{q},$$

where  $g$  is the geometric genus of  $C$ . As a consequence, curves attaining the bound (which are called *maximal*) are particularly interesting in coding theory.

### C. Elliptic curves

Let  $E$  be an elliptic curve over  $\mathbb{F}_q$  and  $\mathcal{O}$  be the point at infinity of  $E(\overline{\mathbb{F}}_q)$ , where  $\overline{\mathbb{F}}_q$  is the algebraic closure of  $\mathbb{F}_q$  and  $E(\overline{\mathbb{F}}_q)$  is the set of all points on  $E$ . For any divisor  $D \in \text{Div}(E)$ , we denote  $\overline{D}$  the unique rational point such that  $D - \overline{D} - (\deg(D) - 1)\mathcal{O}$  is a principal divisor. In fact, if  $D = m_1P_1 + m_2P_2 + \dots + m_nP_n$ , then  $\overline{D} = m_1P_1 \oplus m_2P_2 \oplus \dots \oplus m_nP_n$ , where  $\oplus$  is the addition of points on the elliptic curve. For non-negative integer  $r$ , let  $E[r] := \{P \in E(\overline{\mathbb{F}}_q) : \underbrace{P \oplus \dots \oplus P}_r = \mathcal{O}\}$ . We refer to [22] for more details about elliptic curves.

## III. GENERAL CONSTRUCTIONS OF LCD CODES FROM ELLIPTIC CURVES

In this section, we consider the construction of LCD codes from elliptic curves and determine the parameters of these LCD codes. We first present a proposition, which will be used in the following paper.

*Proposition 3.1:* Let  $a_i \in \mathbb{F}_q^*$  ( $i = 1, \dots, n$ ) and  $\mathcal{C}$  be a linear code in  $\mathbb{F}_q^n$ . If  $\mathcal{C}^\perp = \mathbf{e} \cdot \mathcal{C}'$  with  $\mathbf{e} = (a_1^2, \dots, a_n^2)$  and  $\mathcal{C} \cap \mathcal{C}' = \{0\}$ , then,  $(\mathbf{a} \cdot \mathcal{C})^\perp = \mathbf{a} \cdot \mathcal{C}'$  and  $\mathbf{a} \cdot \mathcal{C}$  is complementary dual, where  $\mathbf{a} = (a_1, \dots, a_n)$ .

*Proof:* From  $\mathcal{C}^\perp = \mathbf{e} \cdot \mathcal{C}'$ , one has  $\dim_{\mathbb{F}_q}(\mathcal{C}) + \dim_{\mathbb{F}_q}(\mathcal{C}') = n$ , and, for any  $(c_1, \dots, c_n) \in \mathcal{C}$  and  $(c'_1, \dots, c'_n) \in \mathcal{C}'$ ,  $a_1c_1 \cdot a_1c'_1 + \dots + a_nc_n \cdot a_nc'_n = 0$ . Thus,  $(\mathbf{a} \cdot \mathcal{C})^\perp = \mathbf{a} \cdot \mathcal{C}'$ .

Suppose  $(a_1\bar{c}_1, \dots, a_n\bar{c}_n) \in (\mathbf{a} \cdot \mathcal{C})^\perp \cap \mathbf{a} \cdot \mathcal{C}$ , where  $(\bar{c}_1, \dots, \bar{c}_n) \in \mathcal{C}$ . Then, for any  $(c_1, \dots, c_n) \in \mathcal{C}$ ,  $a_1c_1 \cdot a_1\bar{c}_1 + \dots + a_nc_n \cdot a_n\bar{c}_n = 0$ . Thus,  $(a_1^2\bar{c}_1, \dots, a_n^2\bar{c}_n) \in \mathcal{C}^\perp$  and  $(\bar{c}_1, \dots, \bar{c}_n) \in \mathcal{C}'$ . From  $\mathcal{C} \cap \mathcal{C}' = \{0\}$ , we obtain  $(\bar{c}_1, \dots, \bar{c}_n) = (0, \dots, 0)$ . Hence,  $\mathbf{a} \cdot \mathcal{C}$  is complementary dual. This completes the proof. ■

The following theorem constructs the LCD codes from elliptic curves and determines the corresponding parameters of these codes.

*Theorem 3.2:* Let  $E$  be an elliptic curve over  $\mathbb{F}_q$  and  $G, D = P_1 + P_2 + \dots + P_n$  be two divisors over  $E$ , where  $0 < \deg(G) < n$ . Let  $\omega$  be a Weil differential such that  $(\omega) = G + H - D$  for some divisor  $H$  and  $\text{Supp}(G) \cap \text{Supp}(D) = \text{Supp}(H) \cap \text{Supp}(D) = \emptyset$ . Assume that

- (1) There is a vector  $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{F}_q^*)^n$  with  $\text{Res}_{P_i}(\omega) = a_i^2$ ;
- (2)  $\deg(\text{g.c.d}(G, H)) = 0$ ;
- (3)  $\text{g.c.d}(G, H)$  is not a principal divisor.

Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $\deg(G)$  and minimum distance  $d \geq n - \deg(G)$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $n - \deg(G)$  and minimum distance  $d^\perp \geq \deg(G)$ .

*Proof:* Note that a canonical divisor over an elliptic curve is a principal divisor. Since  $\text{g.c.d}(G, H)$  is not a principal divisor and  $\text{l.m.d}(G, H) - D = (G + H - D) - \text{g.c.d}(G, H)$ , then  $\text{l.m.d}(G, H) - D$  is not a principal divisor.

We firstly prove that  $\mathcal{C}(D, G) \cap \mathcal{C}(D, H) = \{0\}$ . Suppose that there are some  $f \in \mathcal{L}(G)$  and some  $g \in \mathcal{L}(H)$  such that  $f(P_i) = g(P_i)$  for  $i = \{1, 2, \dots, n\}$ . Consider the following two mutually exclusive cases on  $h := f - g$

- 1) Case  $h = 0$ . One has  $f \in \mathcal{L}(G) \cap \mathcal{L}(H)$ . Then,  $f \in \mathcal{L}(\text{g.c.d}(G, H))$ . Since  $\text{g.c.d}(G, H)$  is not a principal divisor, then  $f \in \mathcal{L}(\text{g.c.d}(G, H)) = \{0\}$  and  $f = g = 0$ .
- 2) Case  $h \neq 0$ . One has  $h \in \mathcal{L}(\text{l.m.d}(G, H) - D)$  as  $h(P_i) = 0$  ( $i = 1, \dots, n$ ). Since  $\text{l.m.d}(G, H) - D$  is not a principal divisor, then  $h \in \mathcal{L}(\text{l.m.d}(G, H) - D) = \{0\}$  and  $h = 0$ , which is a contradiction.

Hence,  $\mathcal{C}(D, G) \cap \mathcal{C}(D, H) = \{0\}$ . From Equation (3),  $\mathcal{C}(D, G)^\perp = \mathbf{e} \cdot \mathcal{C}(D, H)$  with  $\mathbf{e} = (\text{Res}_{P_1}(\omega), \dots, \text{Res}_{P_n}(\omega))$ , and Proposition 3.1,  $\mathcal{GC}(D, G, \mathbf{a})$  and  $\mathcal{GC}(D, H, \mathbf{a})$  are complementary dual codes with  $\mathcal{GC}(D, G, \mathbf{a})^\perp = \mathcal{GC}(D, H, \mathbf{a})$ . The dimensions and minimum distances follow from Theorem 2.2. ■

**Remark** An interesting result of Cheng [2] says that the minimum distance problem is already NP-hard (under RP-reduction) for general elliptic curves codes. In [14], Li et al. showed that the minimum distance of algebraic codes from elliptic curves also has a simple explicit formula if the evaluation set is suitably large (at least  $\frac{2}{3}$  of the group order). This method proves that, if  $n = \#D \geq q + 2$  and  $3 < \deg(G) < q - 1$ , then,  $\mathcal{GC}(D, G, \mathbf{a})$  has the deterministic minimum distance  $n - \deg(G)$ . In this cases,  $\mathcal{GC}(D, G, \mathbf{a})$  has parameters  $[n, \deg(G), n - \deg(G)]$ , where  $n = \#D$ . Thus,  $\mathcal{GC}(D, G, \mathbf{a})$  is an almost MDS code. Let  $n \geq q + 3$ . From [11], if  $2 \leq \deg(G) \leq n - (q + 1)$  or  $q + 1 \leq \deg(G) \leq n - 2$ , the elliptic code  $\mathcal{GC}(D, G, \mathbf{a})$  is optimal. Hence, many different (perhaps nonequivalent) LCD generalized elliptic optimal codes exist, since many elliptic curves with more than  $q + 2$  rational points exist.

From Theorem 3.2, we have the following two corollaries.

*Corollary 3.3:* Let  $n, r$  be positive integers with  $2 \leq r \leq \frac{n+1}{2}$  and  $D = P_1 + \dots + P_n$  be a divisor such that  $\overline{D} \notin E[r - 1]$  and  $\mathcal{O}, \overline{D} \notin \text{Supp}(D)$ . Let  $G = (r - 1)\mathcal{O} + r\overline{D}$  and  $H = (n - r)\mathcal{O} - (r - 1)\overline{D}$ . Let  $\omega$  be the Weil differential such that  $(\omega) = (n - 1)\mathcal{O} + \overline{D} - D$ . Assume that there is a vector  $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{F}_q^*)^n$  with  $\text{Res}_{P_i}(\omega) = a_i^2$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2r - 1$  and minimum distance  $d \geq n - 2r + 1$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $n - 2r + 1$  and minimum distance  $d^\perp \geq 2r - 1$ .

*Proof:* Note that  $G + H - D = (n - 1)\mathcal{O} + \overline{D} - D$  is a principal divisor. Then, there exists a Weil differential  $\omega$  such that  $(\omega) = (n - 1)\mathcal{O} + \overline{D} - D$ . From  $\overline{D} \notin E[r - 1]$ ,  $\text{g.c.d}(G, H) = (r - 1)\mathcal{O} - (r - 1)\overline{D}$  is not a principal divisor. This corollary follows from Theorem 3.2. ■

*Corollary 3.4:* Let  $Q$  be a place on  $E$  different from  $\mathcal{O}$ ,  $G = (r \cdot \deg(Q))\mathcal{O} + rQ$ , and  $D = P_1 + \cdots + P_n$  be a principal divisor, where  $0 < 2r \cdot \deg(Q) < n$ ,  $\underbrace{\overline{Q} \oplus \cdots \oplus \overline{Q}}_r \neq \mathcal{O}$ , and  $Q, \mathcal{O} \notin \text{Supp}(D)$ . Let  $\omega$  be the Weil differential such that

$(\omega) = n\mathcal{O} - D$ . Assume that there is a vector  $\mathbf{a} = (a_1, \cdots, a_n) \in (\mathbb{F}_q^*)^n$  with  $\text{Res}_{P_i}(\omega) = a_i^2$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2r \cdot \deg(Q)$  and minimum distance  $d \geq n - 2r \cdot \deg(Q)$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $n - 2r \cdot \deg(Q)$  and minimum distance  $d^\perp \geq 2r \cdot \deg(Q)$ , where  $H = (n - r \cdot \deg(Q))\mathcal{O} - rQ$ .

*Proof:* Note that  $(\omega) = G + H - D = n\mathcal{O} - D$  and  $\text{g.c.d}(G, H) = (r \cdot \deg(Q))\mathcal{O} - rQ$ . From  $\underbrace{\overline{Q} \oplus \cdots \oplus \overline{Q}}_r \neq \mathcal{O}$ ,  $\text{g.c.d}(G, H)$  is not a principal divisor. The corollary follows from Theorem 3.2.  $\blacksquare$

#### IV. EXPLICIT CONSTRUCTION OF LCD CODES FROM ELLIPTIC CURVES

The previous results presented in Section III are of significance if there are interesting examples of elliptic curves and divisors  $G, H, D$  satisfying the properties assumed in Theorem 3.2. Hence, in this section we present general examples, where the assumptions of Theorem 3.2 are satisfied.

Let  $q = 2^m$  and  $E^{(0)}$  be an elliptic curve defined by the equation

$$y^2 + y = x^3 + bx + c, \quad (4)$$

where  $b, c \in \mathbb{F}_q$ . The point at infinity is denoted by  $\mathcal{O}$ . Let  $S$  be the set of  $x$ -components of the affine points of  $E^{(0)}$  over  $\mathbb{F}_q$ , that is,

$$S := \{\alpha \in \mathbb{F}_q : \text{there is } \beta \in \mathbb{F}_q \text{ such that } \beta^2 + \beta = \alpha^3 + b\alpha + c\}.$$

For any  $\alpha \in S$ , there exactly exist two points with  $x$ -component  $\alpha$ . Denote these two points corresponding to  $\alpha$  by  $P_\alpha^+$  and  $P_\alpha^-$ . Then the set  $E^{(0)}(\mathbb{F}_q)$  of all rational points of  $E^{(0)}$  over  $\mathbb{F}_q$  is  $E^{(0)}(\mathbb{F}_q) = \{P_\alpha^+ : \alpha \in S\} \cup \{P_\alpha^- : \alpha \in S\} \cup \{\mathcal{O}\}$ . The following Lemma can be found in [8].

*Lemma 4.1:* Let  $s$  be a positive integer,  $\{\alpha_1, \cdots, \alpha_s\}$  be a subset of  $S$  with cardinality  $s$ , and  $D = \sum_{i=1}^s (P_{\alpha_i}^+ + P_{\alpha_i}^-)$ . Let  $h = \prod_{i=1}^s (x + \alpha_i)$  and  $\omega = \frac{dx}{h}$ . Then,  $(\omega) = 2s \cdot \mathcal{O} - D$  and

$$\text{Res}_{P_{\alpha_j}^+}(\omega) = \text{Res}_{P_{\alpha_j}^-}(\omega) = \frac{1}{\prod_{i=1, i \neq j}^s (\alpha_j + \alpha_i)},$$

for any  $j \in \{1, 2, \cdots, s\}$ .

The following result is a direct consequence of Corollary 3.4 and Lemma 4.1.

*Theorem 4.2:* Let  $s$  be a positive integer,  $\{\alpha_0, \alpha_1, \cdots, \alpha_s\}$  be a subset of  $S$  with cardinality  $s + 1$ ,  $D = (P_{\alpha_1}^+ + P_{\alpha_1}^-) + \cdots + (P_{\alpha_s}^+ + P_{\alpha_s}^-)$  and  $G = r\mathcal{O} + rP_{\alpha_0}^+$ , where  $0 < r < s$  and  $P_{\alpha_0}^+ \notin E^{(0)}[r]$ . Let  $b_j = \frac{1}{\prod_{i=1, i \neq j}^s (\alpha_j^{2^m-1} + \alpha_i^{2^m-1})}$  ( $j = 1, \cdots, s$ ) and  $\mathbf{a} = (b_1, b_1, \cdots, b_s, b_s)$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2r$  and minimum distance  $d \geq 2(s - r)$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2(s - r)$  and minimum distance  $d^\perp \geq 2r$ , where  $H = (2s - r)\mathcal{O} - rP_{\alpha_0}^+$ .

*Example 1:* Let  $q = 2^4$ ,  $\mathbb{F}_q = \mathbb{F}_2[\rho]$  with  $\rho^4 + \rho + 1 = 0$ , and  $E^{(0)}$  be the elliptic curve defined by  $y^2 + y = x^3 + \rho^3$ . Let  $P^+ = (\rho, 0)$ ,  $P^- = (\rho, 1)$  and  $D = E^{(0)}(\mathbb{F}_q) \setminus \{\mathcal{O}, P^+, P^-\}$ . Then  $4P^+ \neq \mathcal{O}$  and  $\#D = 22$ . Let  $G = 4\mathcal{O} + 4P^+$  and  $H = 18\mathcal{O} - 4P^+$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  in Theorem 4.2 is a LCD code with parameters [22, 8, 14], and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  in Theorem 4.2 of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with parameters [22, 14, 8], which is verified by MAGMA.

*Corollary 4.3:* Let  $N = \#E^{(0)}(\mathbb{F}_q)$  and  $s$  be a positive integer,  $\{\alpha_0, \alpha_1, \cdots, \alpha_s\}$  be a subset of  $S$  with cardinality  $s + 1$ ,  $D = (P_{\alpha_1}^+ + P_{\alpha_1}^-) + \cdots + (P_{\alpha_s}^+ + P_{\alpha_s}^-)$  and  $G = r\mathcal{O} + rP_{\alpha_0}^+$ , where  $0 < r < s$  and  $\text{g.c.d}(r, N) = 1$ . Let  $b_j = \frac{1}{\prod_{i=1, i \neq j}^s (\alpha_j^{2^m-1} + \alpha_i^{2^m-1})}$  ( $j = 1, \cdots, s$ ) and  $\mathbf{a} = (b_1, b_1, \cdots, b_s, b_s)$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2r$  and minimum distance  $d \geq 2(s - r)$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2(s - r)$  and minimum distance  $d^\perp \geq 2r$ , where  $H = (2s - r)\mathcal{O} - rP_{\alpha_0}^+$ .

*Proof:* From  $\text{g.c.d}(r, N) = 1$ , there exist integers  $k_1$  and  $k_2$  such that  $k_1r + k_2N = 1$ . Then,  $(k_1r)P_{\alpha_0}^+ = P_{\alpha_0}^+ \ominus (k_2N)P_{\alpha_0}^+ = P_{\alpha_0}^+ \neq \mathcal{O}$ . Thus,  $P_{\alpha_0}^+ \notin E^{(0)}[r]$ . This corollary follows from Theorem 4.2.  $\blacksquare$

**Remark** The following Table I lists the numbers of rational points of some elliptic curves over  $\mathbb{F}_{2^m}$  [17]. For general elliptic curves over a finite field, we can use the Schoof's algorithms [20] to count the number of rational points.

*Theorem 4.4:* Let  $s$  be a positive integer,  $\{\alpha_1, \cdots, \alpha_s\}$  be a subset of  $S$  with cardinality  $s$ ,  $D = P_{\alpha_1}^- + (P_{\alpha_2}^+ + P_{\alpha_2}^-) + \cdots + (P_{\alpha_s}^+ + P_{\alpha_s}^-)$  and  $G = (r + 1)\mathcal{O} + rP_{\alpha_1}^+$ , such that  $0 \leq r < s - 1$  and  $P_{\alpha_1}^+ \notin E^{(0)}[r + 1]$ . Let  $b_j = \frac{1}{\prod_{i=1, i \neq j}^s (\alpha_j^{2^m-1} + \alpha_i^{2^m-1})}$  ( $j = 1, \cdots, s$ ) and  $\mathbf{a} = (b_1, b_2, b_2, \cdots, b_s, b_s)$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2r + 1$  and minimum distance  $d \geq 2(s - r - 1)$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2(s - r - 1)$  and minimum distance  $d^\perp \geq 2r + 1$ , where  $H = (2s - r - 1)\mathcal{O} - (r + 1)P_{\alpha_1}^+$ .

TABLE I  
THE NUMBERS OF RATIONAL POINTS OF ELLIPTIC CURVES OVER  $\mathbb{F}_q (q = 2^m)$

Elliptic Curve $E^{(0)}$	$m$	$\#E(\mathbb{F}_{2^m})$
$y^2 + y = x^3$	odd $m$	$q + 1$
	$m \equiv 0 \pmod{4}$	$q + 1 - 2\sqrt{q}$
	$m \equiv 2 \pmod{4}$	$q + 1 + 2\sqrt{q}$
$y^2 + y = x^3 + x$	$m \equiv 1, 7 \pmod{8}$	$q + 1 + \sqrt{2q}$
	$m \equiv 3, 5 \pmod{8}$	$q + 1 - \sqrt{2q}$
$y^2 + y = x^3 + x + 1$	$m \equiv 1, 7 \pmod{8}$	$q + 1 - \sqrt{2q}$
	$m \equiv 3, 5 \pmod{8}$	$q + 1 + \sqrt{2q}$
$y^2 + y = x^3 + \delta x \ (\text{Tr}_1^m(\delta) = 1)$	even $m$	$q + 1$
$y^2 + y = x^3 + \omega \ (\text{Tr}_1^m(\omega) = 1)$	$m \equiv 0 \pmod{4}$	$q + 1 + 2\sqrt{q}$
	$m \equiv 2 \pmod{4}$	$q + 1 - 2\sqrt{q}$

*Proof:* From Lemma 4.1,  $(\omega) = G + H - D$ , where  $\omega = \frac{1}{\prod_{i=1}^s (x + \alpha_i)} dx$ . Note that  $\text{g.c.d}(G, H) = (r + 1)\mathcal{O} - (r + 1)P_{\alpha_1}^+$ . This theorem follows from  $P_{\alpha_1}^+ \notin E^{(0)}[r + 1]$  and Theorem 3.2. ■

*Example 2:* Let  $q = 2^4$ ,  $\mathbb{F}_q = \mathbb{F}_2[\rho]$  with  $\rho^4 + \rho + 1 = 0$  and  $E^{(0)}$  be the elliptic curve defined by  $y^2 + y = x^3 + \rho^3$ . Let  $P^+ = (\rho, 0)$  and  $D = E^0(\mathbb{F}_q) \setminus \{\mathcal{O}, P^+\}$ . Then  $4P^+ \neq \mathcal{O}$  and  $\#D = 23$ . Let  $G = 4\mathcal{O} + 3P^+$  and  $H = 20\mathcal{O} - 4P^+$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  in Theorem 4.4 is a LCD code with parameters [23, 7, 16], and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  in Theorem 4.4 of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with parameters [23, 16, 7], which is verified by MAGMA.

*Corollary 4.5:* Let  $N = \#E^{(0)}(\mathbb{F}_q)$  and  $r, s$  be positive integers, where  $0 \leq r < s - 1$  and  $\text{g.c.d}(r + 1, N) = 1$ . Let  $\{\alpha_1, \dots, \alpha_s\}$  be a subset of  $S$  with cardinality  $s$ ,  $D = P_{\alpha_1}^- + (P_{\alpha_2}^+ + P_{\alpha_2}^-) + \dots + (P_{\alpha_s}^+ + P_{\alpha_s}^-)$  and  $G = (r + 1)\mathcal{O} + rP_{\alpha_1}^+$ . Let  $b_j = \frac{1}{\prod_{i=1, i \neq j}^s (\alpha_j^{2^m - 1} + \alpha_i^{2^m - 1})}$  ( $j = 1, \dots, s$ ) and  $\mathbf{a} = (b_1, b_2, b_2 \dots, b_s, b_s)$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2r + 1$  and minimum distance  $d \geq 2(s - r - 1)$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2(s - r - 1)$  and minimum distance  $d^\perp \geq 2r + 1$ , where  $H = (2s - r - 1)\mathcal{O} - (r + 1)P_{\alpha_1}^+$ .

*Proof:* The result follows from Theorem 4.4 and similar arguments used in the proof of Corollary 4.3. ■

*Theorem 4.6:* Let  $r, s$  be integers with  $0 \leq r < \frac{s-2}{2}$ ,  $\{\alpha_0, \alpha_1, \dots, \alpha_s\}$  be a subset of  $S$  with cardinality  $s + 1$ ,  $D = \sum_{i=1}^s (P_{\alpha_i}^+ + P_{\alpha_i}^-)$  and  $G = (2r + 3) \cdot \mathcal{O} + r \cdot (P_{\alpha_0}^+ + P_{\alpha_0}^-) + P_{\alpha_0}^+$ . Let  $b_j = \frac{1}{\prod_{i=0, i \neq j}^s (\alpha_j^{2^m - 1} + \alpha_i^{2^m - 1})}$  ( $j = 1, \dots, s$ ) and  $\mathbf{a} = (b_1, b_1, b_2, b_2 \dots, b_s, b_s)$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $4(r + 1)$  and minimum distance  $d \geq 2s - 4(r + 1)$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $2s - 4(r + 1)$  and minimum distance  $d^\perp \geq 4(r + 1)$ , where  $H = (2s - 2r - 1) \cdot \mathcal{O} - (r + 2) \cdot (P_{\alpha_0}^+ + P_{\alpha_0}^-) + P_{\alpha_0}^-$ .

*Proof:* Let  $h = \prod_{i=0}^s (x + \alpha_i)$  and  $\omega = \frac{1}{h} dx$ . Then, from Lemma 4.1,  $(\omega) = G + H - D$ . Note that  $\text{g.c.d}(G, H) = (2r + 3)\mathcal{O} - (r + 2)(P_{\alpha_0}^+ + P_{\alpha_0}^-) + P_{\alpha_0}^-$ . From  $P_{\alpha_0}^+ \oplus P_{\alpha_0}^- = \mathcal{O}$ ,  $\text{g.c.d}(G, H) \sim P_{\alpha_0}^- - \mathcal{O}$ . Thus,  $\text{g.c.d}(G, H)$  is not a principal divisor. This theorem follows from Theorem 3.2. ■

*Example 3:* Let  $q = 2^4$ ,  $\mathbb{F}_q = \mathbb{F}_2[\rho]$  with  $\rho^4 + \rho + 1 = 0$  and  $E^{(0)}$  be the elliptic curve defined by  $y^2 + y = x^3 + \rho^3$ . Let  $P^+ = (\rho, 0)$ ,  $P^- = (\rho, 1)$  and  $D = E^0(\mathbb{F}_q) \setminus \{\mathcal{O}, P^+, P^-\}$ . Then  $\#D = 22$ . Let  $G = 3\mathcal{O} + P^+$  and  $H = 21\mathcal{O} - 2(P^+ + P^-) + P^-$ . Then,  $\mathcal{GC}(D, G, \mathbf{a})$  in Theorem 4.6 is a LCD code with parameters [22, 4, 18], and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  in Theorem 4.6 of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with parameters [22, 18, 4], which is verified by MAGMA. From the Remark below 3.2, both  $\mathcal{GC}(D, G, \mathbf{a})$  and  $\mathcal{GC}(D, H, \mathbf{a})$  are optimal.

## V. LCD CODES FROM GENERAL ALGEBRAIC CURVES

In this section we consider the construction of LCD codes from any algebraic curves, present a construction mechanism of LCD codes from algebraic geometry codes, and give concrete construction of LCD codes from hyperelliptic curves and Hermitian curves.

Two theorems on constructing LCD codes from algebraic curves are given below.

*Theorem 5.1:* Let  $C$  be a smooth projective curve of genus  $g$  over  $\mathbb{F}_q$  and  $G, D = P_1 + P_2 + \dots + P_n$  be two divisors over  $C$ , where  $2g - 2 < \text{deg}(G) < n$ . Let  $\omega$  be a Weil differential such that  $(\omega) = G + H - D$  for some divisor  $H$  and  $\text{Supp}(G) \cap \text{Supp}(D) = \text{Supp}(H) \cap \text{Supp}(D) = \emptyset$ . Assume that

- (1) There is a vector  $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{F}_q^*)^n$  with  $\text{Res}_{P_i}(\omega) = a_i^2$ ;
- (2)  $\text{g.c.d}(G, H)$  is a non-special divisor of degree  $g - 1$ .

Then,  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $\text{deg}(G) + 1 - g$  and minimum distance  $d \geq n - \text{deg}(G)$ , and the dual code  $\mathcal{GC}(D, H, \mathbf{a})$  of  $\mathcal{GC}(D, G, \mathbf{a})$  is a LCD code with dimension  $n - \text{deg}(G) - 1 + g$  and minimum distance  $d^\perp \geq \text{deg}(G) + 2 - 2g$ .

*Proof:* We first prove that  $\mathcal{C}(D, G) \cap \mathcal{C}(D, H) = \{0\}$ . Suppose that there exist some  $f \in \mathcal{L}(G)$  and  $g \in \mathcal{L}(H)$  such that  $f(P_i) = g(P_i)$  for  $i = \{1, 2, \dots, n\}$ . Consider the following two mutually exclusive cases on  $h := f - g$

1) Case  $h = 0$ . One has  $f \in \mathcal{L}(G) \cap \mathcal{L}(H)$  and  $f \in \mathcal{L}(\text{g.c.d}(G, H))$ . Since  $\text{g.c.d}(G, H)$  is a non-special divisor,  $\mathcal{L}(\text{g.c.d}(G, H)) = \{0\}$  and  $f = g = 0$ .

2) Case  $h \neq 0$ . One has  $h \in \mathcal{L}(l.m.d(G, H) - D)$  as  $h(P_i) = 0 (i = 1, \dots, n)$ . Let  $k = l(1.m.d(G, H) - D) - l(g.c.d(G, H))$ . From  $g.c.d(G, H) = (\omega) - (1.m.d(G, H) - D)$  and Theorem 2.1,

$$\begin{aligned} k &= l(1.m.d(G, H) - D) - l((\omega) - (1.m.d(G, H) - D)) \\ &= deg(1.m.d(G, H) - D) + 1 - g \\ &= deg((\omega) - g.c.d(G, H)) + 1 - g \\ &= 0. \end{aligned}$$

From  $\mathcal{L}(g.c.d(G, H)) = \{0\}$ ,  $\mathcal{L}(1.m.d(G, H) - D) = \{0\}$ . Thus,  $h = 0$ , which is a contradiction.

Hence,  $\mathcal{C}(D, G) \cap \mathcal{C}(D, H) = \{0\}$ . From Equation (3),  $\mathcal{C}(D, G)^\perp = \mathbf{e} \cdot \mathcal{C}(D, H)$  with  $\mathbf{e} = (\text{Res}_{P_1}(\omega), \dots, \text{Res}_{P_n}(\omega))$  and Proposition 3.1,  $\mathcal{GC}(D, G, \mathbf{a})$  and  $\mathcal{GC}(D, H, \mathbf{a})$  are complementary dual codes with  $\mathcal{GC}(D, G, \mathbf{a})^\perp = \mathcal{GC}(D, H, \mathbf{a})$ . The dimensions and minimum distances follow from Theorem 2.2. ■

**Remark** In [1], S. Ballet and D. Le Brigand proved that if  $\#C(\mathbb{F}_q) \geq g + 1$ , there exists a non-special divisor such that  $deg(G) = g - 1$  and  $Supp(G) \subset C(\mathbb{F}_q)$ . Then, the existence of non-special divisors of degree  $g - 1$  is often clear since the involved algebraic curves have many rational points. However, the problem lies in their effective determination. Moreover, actually almost all the divisors with degree  $g - 1$  are non-special (the terminology almost all means all but finitely many).

*Example 4:* Let  $q = 2$  and  $C$  be the projective curve of genus 1 defined by  $Y^2Z + YZ^2 = X^3$  over  $\mathbb{F}_4 = \{0, 1, \rho, \rho^2\}$ . Then,  $C(\mathbb{F}_4) = \{\mathcal{O}, Q, P_1, \dots, P_7\} = \{(0 : 1 : 0), (0 : 0 : 1), (0 : 1 : 1), (\rho : \rho : 1), (\rho : \rho^2 : 1), (\rho^2 : \rho : 1), (\rho^2 : \rho^2 : 1), (1 : \rho : 1), (1 : \rho^2 : 1)\}$ . Let  $D = \{P_1, \dots, P_7\}$ ,  $G = 2\mathcal{O} + Q$  and  $H = 6\mathcal{O} - 2Q$ . Then,  $g.c.d(G, H) = 2\mathcal{O} - 2Q$  is non-special,  $(\frac{Z^4}{X^4+Z^3X} d\frac{X}{Z}) = G + H - D$  and  $\text{Res}_{P_i}(\frac{Z^4}{X^4+Z^3X} d\frac{X}{Z}) = 1$  for  $i \in \{1, \dots, 7\}$ . Note that  $(\frac{X}{Z}) = Q - 2\mathcal{O} + P_1$  and  $(\frac{Y+Z}{X}) = -Q - \mathcal{O} + 2P_1$ . Thus,  $\{1, \frac{X}{Z}, \frac{Y+Z}{X}\}$  is a basis of  $\mathcal{L}(2\mathcal{O} + Q)$ . Evaluate the functions in  $\{1, \frac{X}{Z}, \frac{Y+Z}{X}\}$  at the places  $\{P_1, \dots, P_7\}$ . One obtains the generator matrix of  $\mathcal{C}(D, G)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \rho & \rho & \rho^2 & \rho^2 & 1 & 1 \\ 0 & \rho & 1 & 1 & \rho^2 & \rho^2 & \rho \end{pmatrix}$$

Moreover,  $\mathcal{C}(D, G)$  is a LCD code with parameters [7, 3, 4], which is verified by MAGMA. This code is optimal.

*Theorem 5.2:* Let  $C$  be a smooth projective curve of genus  $g$  over  $\mathbb{F}_q$  and  $G, D = P_1 + P_2 + \dots + P_n$  be two divisors over  $C$ , where  $2g - 2 < deg(G) < n$ . Let  $\omega$  be a Weil differential such that  $(\omega) = G + H - D$  for some divisor  $H$  and  $Supp(G) \cap Supp(D) = Supp(H) \cap Supp(D) = \emptyset$ . Assume that

- (1)  $\text{Res}_{P_i}(\omega) = \text{Res}_{P_j}(\omega)$  for  $1 \leq i < j \leq n$ ;
- (2)  $g.c.d(G, H)$  is a divisor of degree  $g - 1$ .

Then,  $\mathcal{C}(D, G)$  is a LCD code if and only if  $g.c.d(G, H)$  is non-special.

*Proof:* Let  $\text{Res}_{P_i} = c$  for  $1 \leq i \leq n$ . From Equation (3),  $\mathcal{C}(D, G)^\perp = (c, \dots, c) \cdot \mathcal{C}(D, H) = \mathcal{C}(D, H)$ .

Suppose that  $g.c.d(G, H)$  is non-special.  $\mathcal{C}(D, G)$  is a LCD code from similar arguments used in proving Theorem 5.1.

Suppose that  $\mathcal{C}(D, G)$  is a LCD code. If  $g.c.d(G, H)$  is special. Then,  $l(g.c.d(G, H)) > 0$ . Let  $0 \neq f \in \mathcal{L}(g.c.d(G, H)) = \mathcal{L}(G) \cap \mathcal{L}(H)$ . Thus,  $(f(P_1), \dots, f(P_n)) \in \mathcal{C}(D, G) \cap \mathcal{C}(D, H)$ . Note that  $(f(P_1), \dots, f(P_n)) \neq (0, \dots, 0)$ , which contradicts  $\mathcal{C}(D, G) \cap \mathcal{C}(D, H) = \{0\}$ . This completes the proof. ■

*Corollary 5.3:* Let  $C$  be the projective line over  $\mathbb{F}_q$ ,  $\mathcal{O}$  be the point at infinity,  $P$  be the original point and  $D = C(\mathbb{F}_q) \setminus \{\mathcal{O}, P\}$ . Let  $G = r\mathcal{O} + rP$  with  $0 < r \leq \frac{q-2}{2}$ . Then,  $\mathcal{C}(D, G)$  is a maximum distance separable (MDS) LCD code over  $\mathbb{F}_q$  with parameters  $[q - 1, 2r + 1, n - 2r]$ . Moreover,  $\mathcal{C}(D, G)$  has generator matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \rho^1 & \rho^2 & \dots & \rho^{q-2} \\ 1 & \rho^{-1} & \rho^{-2} & \dots & \rho^{-(q-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho^{i-1} & \rho^{i-2} & \dots & \rho^{i-(q-2)} \\ 1 & \rho^{-i-1} & \rho^{-i-2} & \dots & \rho^{-i-(q-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho^{r-1} & \rho^{r-2} & \dots & \rho^{r-(q-2)} \\ 1 & \rho^{-r-1} & \rho^{-r-2} & \dots & \rho^{-r-(q-2)} \end{pmatrix},$$

where  $\rho \in \mathbb{F}_q$  is a primitive element.

*Proof:* Let  $H = (q - r - 2)\mathcal{O} - (r + 1)P$ . Then,  $g.c.d = r\mathcal{O} - (r + 1)P$  is non-special and  $(\omega) = G + H - D$  with  $\omega = \frac{1}{x^q - x} dx$ . Note that  $\text{Res}_Q(\omega) = -1$  for any  $Q \in Supp(D)$ . Form Theorem 5.2,  $\mathcal{C}(D, G)$  is a LCD code. The parameters of  $\mathcal{C}(D, G)$  follows from Theorem 2.2 and the Singleton bound. Observe that  $\{x^i : -r \leq i \leq r\}$  is a base of  $\mathcal{L}(r\mathcal{O} + rP)$ . This completes the proof. ■

### A. LCD codes from hyperelliptic curves

Let  $q = 2^m$  and  $C$  be the curve over  $\mathbb{F}_{q^2}$  defined as

$$y^2 + y = x^{q+1}.$$

This curve has genus  $g = \frac{q}{2}$ . For any  $\alpha \in \mathbb{F}_{q^2}$ , there exactly exist two rational points  $P_\alpha^+, P_\alpha^-$  with  $x$ -component  $\alpha$ . Let  $\mathcal{O}$  be the point at infinity. Then, the set  $C(\mathbb{F}_{q^2})$  of all rational points of  $C$  equal  $\{P_\alpha^+ : \alpha \in \mathbb{F}_{q^2}\} \cup \{P_\alpha^- : \alpha \in \mathbb{F}_{q^2}\} \cup \{\mathcal{O}\}$ . Thus,  $C$  has exactly  $1 + 2q^2 = 1 + q^2 + 2g\sqrt{q^2}$  rational points, which attains the well-known Hasse-Weil bound. Let  $\omega = \frac{1}{x^{q^2+x}}dx$ , then

$$(\omega) = 2(q^2 - 1 + \frac{q}{2}) \cdot \mathcal{O} - \sum_{\alpha \in \mathbb{F}_{q^2}} (P_\alpha^+ + P_\alpha^-) \text{ and } \text{Res}_{P_\alpha^+}(\omega) = \text{Res}_{P_\alpha^-}(\omega) = 1, \quad (5)$$

for any  $\alpha \in \mathbb{F}_{q^2}$ .

**Theorem 5.4:** Let  $q \geq 4$ ,  $P$  be an affine point of  $C$ ,  $D = C(\mathbb{F}_{q^2}) \setminus \{\mathcal{O}, P\}$  and  $G = (r + \frac{q}{2})\mathcal{O} + rP$ , such that  $\frac{q}{4} \leq r \leq q^2 - \frac{q}{4} - 1$  and  $(r + \frac{q}{2})\mathcal{O} - (r + 1)P$  is a non-special divisor. Then,  $\mathcal{C}(D, G)$  is a LCD code with dimension  $2r + 1$  and minimum distance  $d \geq 2q^2 - \frac{q}{2} - 2r - 1$ , and the dual code  $\mathcal{C}(D, H)$  of  $\mathcal{C}(D, G)$  is a LCD code with dimension  $2(q^2 - r - 1)$  and minimum distance  $d^\perp \geq 2r - \frac{q}{2} + 2$ , where  $H = (2q^2 + \frac{q}{2} - r - 2)\mathcal{O} - (r + 1)P$ .

*Proof:* From Equation (5),  $(\omega) = G + H - D$ , where  $\omega = \frac{1}{x^{q^2+x}}dx$ . Observe that  $\text{g.c.d}(G, H) = (r + \frac{q}{2})\mathcal{O} - (r + 1)P$ . This theorem follows from  $(r + \frac{q}{2})\mathcal{O} - (r + 1)P$  being a non-special divisor and Theorem 5.1.  $\blacksquare$

**Example 5:** Let  $q = 2^2$  and  $C$  be the genus 2 hyperelliptic curve defined by  $y^2 + y = x^{q+1}$  over  $\mathbb{F}_{q^2}$ . Let  $P = (0, 0)$  and  $D = C(\mathbb{F}_{q^2}) \setminus \{\mathcal{O}, P\}$ . Then  $9\mathcal{O} - 8P$  is non-special and  $\#D = 31$ . Let  $G = 9\mathcal{O} + 7P$  and  $H = 25\mathcal{O} - 8P$ . Then,  $\mathcal{C}(D, G)$  in Theorem 5.4 is a LCD code with parameters [31, 15], and the dual code  $\mathcal{C}(D, H)$  in Theorem 5.4 of  $\mathcal{C}(D, G)$  is a LCD code with parameters [31, 16], which is verified by MAGMA.

**Theorem 5.5:** Let  $q \geq 4$ ,  $\sum_{i=1}^t n_i = g$  with  $n_i > 0$  and  $\sum_{i=1}^t r_i \leq \frac{1}{4}(2q^2 - \frac{3}{2}q - 4t - 4)$  with  $r_i \geq 0$ . Let  $\{\alpha_1, \dots, \alpha_t\}$  be a subset of  $\mathbb{F}_{q^2}$  with cardinality  $t$ ,  $D = C(\mathbb{F}_{q^2}) \setminus \{\mathcal{O}, P_{\alpha_1}^+, P_{\alpha_1}^-, \dots, P_{\alpha_t}^+, P_{\alpha_t}^-\}$  and  $G = (2(t + \sum_{i=1}^t r_i) + q - 1) \cdot \mathcal{O} + \sum_{i=1}^t r_i \cdot (P_{\alpha_i}^+ + P_{\alpha_i}^-) + \sum_{i=1}^t n_i \cdot P_{\alpha_i}^+$ . Then,  $\mathcal{C}(D, G)$  is a LCD code with dimension  $4 \sum_{i=1}^t r_i + 2t + q$  and minimum distance  $d \geq 2q^2 - 4 \sum_{i=1}^t r_i - 4t - \frac{3}{2}q + 1$ , and the dual code  $\mathcal{C}(D, H)$  of  $\mathcal{C}(D, G)$  is a LCD code with dimension  $2q^2 - 4 \sum_{i=1}^t r_i - 4t - q$  and minimum distance  $d^\perp \geq 4 \sum_{i=1}^t r_i + 2t + \frac{1}{2}q + 1$ , where  $H = (2q^2 - 2(t + \sum_{i=1}^t r_i) - 1) \cdot \mathcal{O} - \sum_{i=1}^t (r_i + n_i + 1) \cdot (P_{\alpha_i}^+ + P_{\alpha_i}^-) + \sum_{i=1}^t n_i \cdot P_{\alpha_i}^-$ .

*Proof:* From Equation (5),  $(\omega) = G + H - D$ , where  $\omega = \frac{1}{x^{q^2+x}}dx$ . Note that  $\text{g.c.d}(G, H) = (2(t + \sum_{i=1}^t r_i) + q - 1) \cdot \mathcal{O} - \sum_{i=1}^t (r_i + n_i + 1) \cdot (P_{\alpha_i}^+ + P_{\alpha_i}^-) + \sum_{i=1}^t n_i P_{\alpha_i}^-$ . From  $P_{\alpha_i}^+ + P_{\alpha_i}^- \sim 2\mathcal{O}$ , we have  $\text{g.c.d}(G, H) = \sum_{i=1}^t n_i P_{\alpha_i}^- - \mathcal{O}$ . Since  $\sum_{i=1}^t n_i P_{\alpha_i}^-$  is a reduced divisor,  $l(\sum_{i=1}^t n_i P_{\alpha_i}^-) = 1$ . Thus  $l(\text{g.c.d}(G, H)) = l(\sum_{i=1}^t n_i P_{\alpha_i}^- - \mathcal{O}) = 0$ . This theorem follows from Theorem 5.1.  $\blacksquare$

**Remark** From  $(x + \alpha_i) = P_{\alpha_i}^+ + P_{\alpha_i}^- - 2\mathcal{O}$ ,  $G = (\prod_{i=1}^t (x + \alpha_i)^{r_i}) + (4 \sum_{i=1}^t r_i + 2t + q - 1) \cdot \mathcal{O} + \sum_{i=1}^t n_i P_{\alpha_i}^+$ . Thus,  $\mathcal{L}(G) = \frac{1}{\prod_{i=1}^t (x + \alpha_i)^{r_i}} \mathcal{L}((4 \sum_{i=1}^t r_i + 2t + q - 1) \cdot \mathcal{O} + \sum_{i=1}^t n_i P_{\alpha_i}^+)$ . From the similar discussion as above, one gets  $\mathcal{L}(H) = \prod_{i=1}^t (x + \alpha_i)^{r_i + n_i + 1} \mathcal{L}((2q^2 - q - \sum_{i=1}^t r_i - 4t - 1) \cdot \mathcal{O} + \sum_{i=1}^t n_i P_{\alpha_i}^-)$ . Let  $D = \{P_{\beta_1}^+, P_{\beta_1}^-, \dots, P_{\beta_{q^2-t}}^+, P_{\beta_{q^2-t}}^-\}$ . Then,  $\mathcal{C}(D, G) = \mathcal{GC}(D, (4 \sum_{i=1}^t r_i + 2t + q - 1) \cdot \mathcal{O} + \sum_{i=1}^t n_i P_{\alpha_i}^+, \mathbf{a})$  and  $\mathcal{C}(D, H) = \mathcal{GC}(D, (2q^2 - q - \sum_{i=1}^t r_i - 4t - 1) \cdot \mathcal{O} + \sum_{i=1}^t n_i P_{\alpha_i}^-, \mathbf{b})$ , where  $\mathbf{a} = (a_1, a_1, \dots, a_{q^2-t}, a_{q^2-t})$  and  $\mathbf{b} = (b_1, b_1, \dots, b_{q^2-t}, b_{q^2-t})$ ,  $a_j = \frac{1}{\prod_{i=1}^t (\beta_j + \alpha_i)^{r_i}}$ , and  $b_j = \prod_{i=1}^t (\beta_j + \alpha_i)^{r_i + n_i + 1}$ .

**Example 6:** Let  $q = 2^3$  and  $C$  be the genus 4 hyperelliptic curve defined by  $y^2 + y = x^{q+1}$  over  $\mathbb{F}_{q^2}$ . Let  $P^+ = (0, 0)$ ,  $P^- = (0, 1)$  and  $D = C(\mathbb{F}_{q^2}) \setminus \{\mathcal{O}, P^+, P^-\}$ . Then  $\#D = 126$ . Let  $G = 19\mathcal{O} + 5(P^+ + P^-) + 4P^+$  and  $H = 115\mathcal{O} - 10(P^+ + P^-) + P^-$ . Then,  $\mathcal{C}(D, G)$  in Theorem 5.5 is a LCD code with parameters [126, 30], and the dual code  $\mathcal{C}(D, H)$  in Theorem 5.5 of  $\mathcal{C}(D, G)$  is a LCD code with parameters [126, 96], which is verified by MAGMA.

**Remark** All constructions presented in Theorem 5.4 and Theorem 5.5 can be directly generalized to any hyperelliptic curves.

### B. LCD codes from Hermitian curves

Let  $q$  be a power of any prime and  $C_{as}$  be the Hermitian curve over  $\mathbb{F}_{q^2}$  defined by

$$y^q + y = x^{q+1}.$$

Then  $C_{as}$  is also an Artin-Schreier curve. The curve  $C_{as}$  has genus  $g = \frac{1}{2}q(q-1)$ , and for every  $\alpha \in \mathbb{F}_{q^2}$  the element  $x - \alpha$  has  $q$  zeros of degree one in  $C_{as}$ . Except the point  $\mathcal{O}$  at infinity, all rational points of  $C_{as}$  are obtained in this way. One easily checks that the Hasse-Weil bound is attained. Let  $\omega = \frac{1}{x^{q^2-x}}dx$ , then

$$(\omega) = (n + 2g - 2)\mathcal{O} - D,$$

where  $D = C_{as}(\mathbb{F}_{q^2}) \setminus \{\mathcal{O}\}$  and  $n = \#D = q^3$ . Then,  $\text{Res}_P(\omega) = -1$ . We refer to [23] for more details about Hermitian curves.

*Theorem 5.6:* Let  $C_{ad}, g, \mathcal{O}, D$  and  $\omega$  be defined as before. Let  $G = (r \cdot \text{deg}(P) + g - 1) \cdot \mathcal{O} + r \cdot P$ , where  $P$  is a place of  $C_{as}$  with degree more than 1 and  $r$  is a positive integer with  $r \cdot \text{deg}(P) \leq \frac{n}{2}$ . Then,  $\mathcal{C}(D, G)$  is a LCD codes if and only if  $(r \cdot \text{deg}(P) + g - 1) \cdot \mathcal{O} - r \cdot P$  is non-special.

*Proof:* Let  $H = (n - r \cdot \text{deg}(P) + g - 1) \cdot \mathcal{O} - r \cdot P$ . Then  $\text{g.c.d}(G, H) = (r \cdot \text{deg}(P) + g - 1) \cdot \mathcal{O} - r \cdot P$ ,  $(\omega) = G + H - D$  and  $\text{Res}_Q(\omega) = -1$  for any  $Q \in \text{Supp}(D)$ . From Theorem 5.2, this theorem follows. ■

*Example 7:* Let  $q = 3$  and  $C$  be the genus 3 Hermitian curve defined by  $y^q + y = x^{q+1}$  over  $\mathbb{F}_{q^2}$ . Let  $P$  be the degree 3 place at  $(\beta, \rho^2\beta^2 + \beta - 1)$ , where  $\rho \in \mathbb{F}_9, \beta \in \mathbb{F}_{9^3}, \rho^2 - \rho - 1 = 0$ , and  $\beta^3 + \rho\beta^2 - \beta + \rho^2 = 0$ . Let  $D = C(\mathbb{F}_{q^2}) \setminus \{\mathcal{O}\}$  and  $G = 8\mathcal{O} + 2P$ . Then,  $\#D = 27$  and  $8\mathcal{O} - 2P$  is non-special.  $\mathcal{C}(D, G)$  in Theorem 5.6 is a LCD code with parameters [27, 12], which is verified by MAGMA.

## VI. CONCLUSION

This paper is devoted to the construction of particular AG complementary dual (LCD) codes which can be resistant against side-channel attacks (SCA). We firstly provide a construction scheme for obtaining LCD codes from elliptic curves and present some explicit LCD codes from elliptic curves, which contain some infinite class of optimal codes with parameters meeting Griesmer bound on linear codes. All codes constructed from elliptic curve are MDS or almost MDS. We also introduce a construction mechanism for obtaining LCD codes from any algebraic curve and derive some explicit LCD codes from hyperelliptic curves and Hermitian curves. In a future work, we will study the resistance of algebraic geometry LCD codes to SCA.

## REFERENCES

- [1] S. Ballet and D. Le Brigand, On the existence of non-special divisors of degree  $g$  and  $g - 1$  in algebraic function fields over  $\mathbb{F}_q$ , Journal of Number Theory, 116, 293-310, 2006.
- [2] Q. Cheng, Hard problems of algebraic geometry codes. IEEE Transactions on Information Theory, 54, pp. 402-406, 2008.
- [3] C. Carlet and S. Guilley, Complementary dual codes for counter-measures to side-channel attacks, In: E. R. Pinto et al. (eds.), Coding Theory and Applications, CIM Series in Mathematical Sciences, vol. 3, pp. 97-105, Springer Verlag, 2014.
- [4] S. M. Dodunekov, Minimum block length of a linear  $q$ -ary code with specified dimension and code distance, Prob. Inform. Transmission 20, pp. 239-249, 1985.
- [5] C. Ding, Chengju Li, and Shuxing Li, LCD Cyclic Codes over Finite Fields, arXiv:1608.0217v1 [cs.IT].
- [6] S. T. Dougherty, J.-L. Kim, B. Ozkaya, L. Sok and P. Solé, The combinatorics of LCD codes: Linear Programming bound and orthogonal matrices, arXiv:1506.01955v1.
- [7] M. Esmaeili and S. Yari, On complementary-dual quasi-cyclic codes, Finite Fields and Their Applications, vol. 15, pp. 375-386, 2009.
- [8] H. Hasse, Theorie der relativ zyklischen algebraischen funktionenkörper, J. Reine Angew. Math., 172, pp. 37-54, 1934.
- [9] N. Hamada and M. Deza, Characterization of  $\{2(q+1)+2, 2; t, q\}$ -min-hyper in  $\text{PG}(t, q)$  ( $t \geq 3, q \geq 5$ ) and its applications to error-correcting codes, Discr. Math., 71, pp. 219-231, 1988.
- [10] N. Hamada and F. Tamari, Constructions of optimal linear codes using flats and spreads in finite projective geometry, Europ. J. Combin. 3, pp. 129-141, 1982.
- [11] H. Janwa, Some optimal codes from algebraic geometry and their covering radii, Europ. J. Combinatorics 11, pp. 249-266, 1990.
- [12] S. Li, C. Ding, and H. Liu, A family of reversible BCH codes, arXiv:1608.02169v1 [cs.IT].
- [13] S. Li, C. Ding, and H. Liu, Parameters of two classes of LCD BCH codes, arXiv:1608.02670 [cs.IT].
- [14] J. Li, D. Wan, and J. Zhang, On the minimum distance of elliptic curve codes, in Information Theory (ISIT), 2015 IEEE International Symposium on, pp. 2391-2395, 2015.
- [15] J. L. Massey, Reversible codes, Information and Control, vol. 7, no. 3, pp. 369-380, 1964.
- [16] J. L. Massey, Linear codes with complementary duals, Discrete Math., vol. 106/107, pp. 337-342, 1992.
- [17] A. Menezes, Elliptic Curve Public Key Cryptosystems, Kluwer Academic Publishers, 1993.
- [18] S. K. Muttoo and S. Lal, A reversible code over  $\text{GF}(q)$ , Kybernetika, vol. 22, pp. 85-91, 1986.
- [19] K. K. Tzeng and C. R. P. Hartmann, On the minimum distance of certain reversible cyclic codes, IEEE Trans. Inform. Theory, vol.16, pp. 644-646, 1970.
- [20] R. Schoof, Counting points on elliptic curves over finite fields, Journal de theorie des nombres de Bordeaux, 7, pp. 219-254, 1995.
- [21] N. Sendrier, Linear codes with complementary duals meet the Gilbert-Varshamov bound, Discrete Mathematics, vol. 285, pp. 345-347, 2004.
- [22] J. H. Silverman, The Arithmetic of Elliptic Curves, GTM106, New York Heidelberg Berlin: Springer-Verlag, 1985.
- [23] H. Stichtenoth, Algebraic function fields and codes, Springer, 2008.
- [24] X. Yang, J. L. Massey, The necessary and sufficient condition for a cyclic code to have a complementary dual, Discrete Math., vol.126, pp. 391-393, 1994.